Review of solutions of linear systems (2-variable)

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

a_i, b_i, c_i given/known
x, y unknown

$$\begin{cases} \boxed{a_1 x + b_1 y = c_1} & \text{"}R_1\text{"} \\ \boxed{a_2 x + b_2 y = c_2} & \text{"}R_2\text{"} \end{cases}$$

for "Row 1." Term/Notation relates to matrices.
("Eq 1" would also be fine)

A solution to the system is a pair (x, y) which makes both equalities hold.

Ex: Solve the system

$$\begin{cases} x + 2y = 0 & (R_1) \\ 2x - y = 5 & (R_2) \end{cases}$$

Step 1 Eliminate x from R_2 by adding $(R_2) + (-2)\cdot(R_1)$

$$\begin{array}{rl} 2x - y = 5 & R_2 \\ + (-2)(x + 2y = 0) & + (-2)(R_1) \\ \hline \Rightarrow \quad 0x - 5y = 5. & \end{array}$$

Solve for y $\longrightarrow$ $y = -1.$

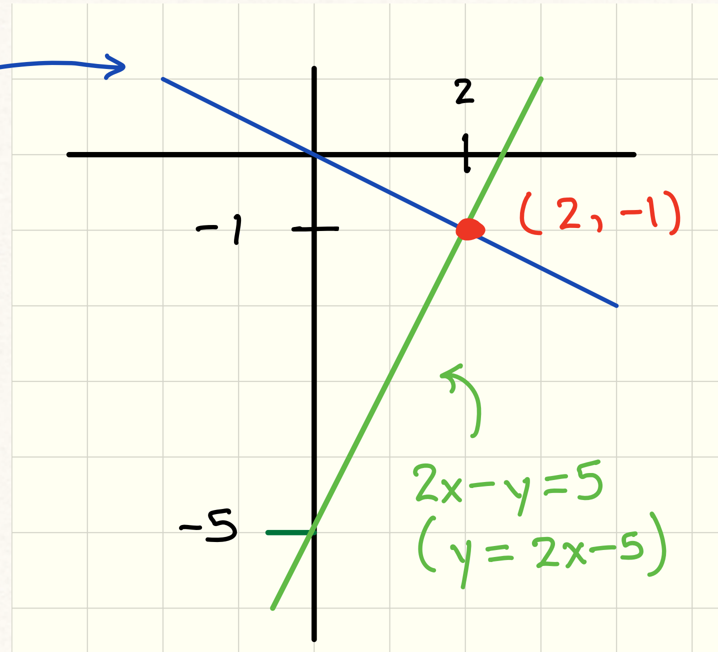
Step 2 Substitute $y = -1$ into R_1: $x + 2(-1) = 0$
$\Rightarrow x - 2 = 0$
$\Rightarrow x = 2$

Thus, $(x, y) = (2, -1)$ is a solution to the system. In fact, it is the unique solution.

Graphically,

Linear eq. $x + 2y = 0 \longleftrightarrow$ line $y = -\frac{1}{2}x$
" " $2x - y = 5 \longleftrightarrow$ line $y = 2x - 5$

$x + 2y = 0$
$(y = -\frac{1}{2}x)$



solution $(x, y) = (2, -1)$
is point where lines
intersect

In manipulating systems of linear eqs,

$$\left(\text{say} \quad \begin{cases} x + 2y = 0 & (R_1) \\ 2x - y = 5 & (R_2) \end{cases} \quad \text{again for example} \right)$$

there are 3 "operations" we can do to the system — after each operation we "reset" the labels of the rows/equations to be $R_1, R_2, \cdots$ (see examples below):

1) Swap two rows: $(R_i \longleftrightarrow R_j)$

$$\begin{cases} x + 2y = 0 & (R_1) \\ 2x - y = 5 & (R_2) \end{cases}$$

operation $(R_1 \longleftrightarrow R_2)$ changes system to

$$\begin{cases} 2x - y = 5 & (R_1) \\ x + 2y = 0 & (R_2) \end{cases}$$

(note $2x - y = 5$ is the "new R_1")

2) Scale an individual row by multiplying it by some number ("scalar") c : (cR_i)

Ex
$$\begin{cases} x + 2y = 0 \\ 6x - 3y = 15 \end{cases}$$

$$\left(\tfrac{1}{3}R_2\right) \begin{cases} x + 2y = 0 \\ 2x - y = 5 \end{cases}$$ is the new system

As far as what the solutions (if any) to the system are, certainly these two operations (swapping and scaling) won't change the sols. In fact our third operation also will not change what the solutions to a system are:

3) Add a multiple of one row to another :

($R_j + (c)R_i$) (R_j is the row that changes)

Ex (what we did in the very first example)
$$\begin{cases} x + 2y = 0 \\ 2x - y = 5 \end{cases}$$

$$(R_2 + (-2)R_1) \begin{cases} x + 2y = 0 \\ 0 - 5y = 5 \end{cases}$$

The 3 operations can be used on systems of linear eqs having more than 2 variables and/or rows :

Ex :
$$\begin{cases} x + 2y - z = 0 & (R_1) \\ 2x - 3y + z = 7 & (R_2) \\ -x + y + 2z = 3 & (R_3) \end{cases}$$

★ A system of <u>linear</u> eqs (in any number of variables) must have either:

1) no solutions, (system is "<u>inconsistent</u>")
2) a <u>single</u>, unique solution,
3) infinitely-many solutions. } (system is "<u>consistent</u>")

Ex: Find all (if any) solutions to the system

$$\begin{cases} x + y = 1 & (R_1) \\ 2x + 2y = 3 & (R_2) \end{cases}$$

Add $(R_2 + (-2)R_1)$ $\begin{cases} x + y = 1 \\ \underline{0 + 0 = 1} \end{cases} \Rightarrow 0 = 1$

Trying to solve the original system led to a system having a contradiction/ impossible eq. $(0 = 1)$

Thus, the original system must have <u>no solutions</u> to begin with.

Alternatively, with the original system

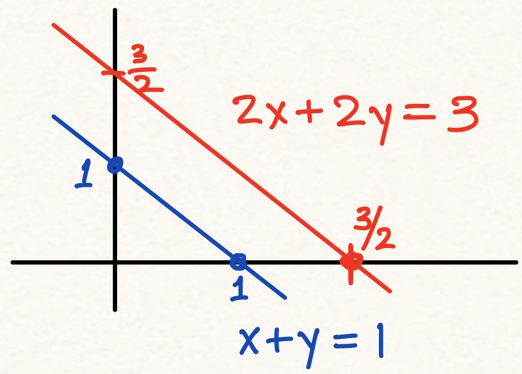
$$\begin{cases} x + y = 1 \\ 2x + 2y = 3 \end{cases}$$

we could do

$$\left(\tfrac{3}{2}R_2\right) \begin{cases} x + y = 1 \\ x + y = 9/2 \end{cases}$$

certainly $x+y$ cannot $= 1$ <u>and</u> $= 3/2$ simultaneously

Graphically: Lines do not intersect (they are parallel but non-overlapping)



Using our other term, the system

$$\begin{cases} x+y=1 \\ 2x+2y=3 \end{cases}$$ is <u>inconsistent</u>.

<u>Ex</u> Determine the solutions (if any) to the system

$$\begin{cases} 2x+6y=4 & (R_1) \\ 3x+9y=6 & (R_1) \end{cases}$$

Scale $(\frac{1}{2}R_1)$ $\begin{cases} x+3y=2 \\ 3x+9y=6 \end{cases}$

Add $(R_2+(-3)R_1)$ $\begin{cases} x+3y=2 & (R_1) \\ 0=0 & (R_2) \end{cases}$

Thus, the system is "one" eq in disguise. There are <u>infinitely-many solutions</u>: Indeed, our R_2 is a "freebie" because certainly $0=0$, and so we only have R_1:

$$x+3y=2 \Rightarrow \underline{x=2-3y}$$

Thus, if we let t be <u>any</u> real number (t is basically a "<u>free parameter</u>")

and set $\begin{cases} \underline{x=2-3y=2-3t} \\ \underline{y=t} \end{cases}$,

then we get a different (for each t) solution for the system.

For instance,

$$\left.\begin{aligned} t=0 &\rightarrow (2,0) \\ t=1 &\rightarrow (-1,1) \\ t=-7 &\rightarrow (23,-7) \\ t=1/2 &\rightarrow \left(\tfrac{1}{2},\tfrac{1}{2}\right) \\ &\cdots \end{aligned}\right\} \text{these are all solutions } (x,y) \text{ for system.}$$

Graphically,

Line $2x+6y=4$

and line $3x+9y=6$

overlap entirely